

# Influence of general convective motions on the exterior of isolated rotating bodies in equilibrium

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**Abstract.** The problem of describing isolated rotating bodies in equilibrium in General Relativity has so far been treated under the assumption of the circularity condition in the interior of the body. For a fluid without energy flux, this condition implies that the fluid flow moves only along the angular direction, i.e. there is no convection. Using this simplification, some recent studies have provided us with uniqueness and existence results for asymptotically flat vacuum exterior fields given the interior sources. Here, the generalisation of the problem to include general sources is studied. It is proven that the convective motions have no direct influence on the exterior field, and hence, that the aforementioned results on uniqueness and existence of exterior fields apply equally in the general case.

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## 1. Introduction

The theory for astrophysical self-gravitating isolated, rotating bodies in equilibrium is still poorly understood in General Relativity (GR). In particular, there is no known explicit global model describing the interior of a rotating object of a finite size in stationary regime (equilibrium) together with its exterior. This fact has driven important efforts onto the finding of theoretical general results for global models suitable for those purposes. Apart from stationarity, and since we are interested in rotating objects, axial symmetry is a natural assumption, and thus we will be interested in spacetimes admitting a global two-dimensional Abelian group of isometries ( $G_2$ ) [1] acting on timelike surfaces. In addition, and to account for the isolation of the body, the model is also required to be asymptotically flat.

From the theoretical point of view, the most usual way to attack the global problem consists in dividing the spacetime into two regions, an *interior* region  $(\mathcal{V}_I, g_I)$ , devised to describe our spatially compact object, and a vacuum and asymptotically flat *exterior* region  $(\mathcal{V}_E, g_E)$ . Both regions are separated by the history of the limiting surface of the body  $\Sigma$ . Then, corresponding Einstein field equations apply at the interior and exterior sides taking common boundary data at  $\Sigma$ . In short, the global problem is divided into two, namely the interior and the exterior problems. They can be treated

independently by looking for existence and uniqueness results on the spacetimes with boundary  $(\mathcal{V}_I, g_I)$  and  $(\mathcal{V}_E, g_E)$  for given data on their respective boundaries  $\Sigma^I$  and  $\Sigma^E$ . Eventually, the two problems have to be ‘matched’ so that the two boundaries  $\Sigma^I$  and  $\Sigma^E$  can be identified as  $\Sigma$ . Therefore, one has to look for compatible data on  $\Sigma^I$  and  $\Sigma^E$  through the imposition of the matching conditions.

Regarding the exterior problem, existence and uniqueness for a stationary axially symmetric asymptotically flat vacuum solution for a given interior are under intense current investigation [2, 3, 4]‡. Under the simplifying assumption of non-convective motions in the interior region (see below), uniqueness was solved in [2] and necessary conditions on the interior for the existence of the exterior have been already found [4]. These conditions turn out to be also sufficient for static exteriors, and thus it has been conjectured to be also true for the stationary ones [4]. This issue is still under investigation.

A well known fact is that the Abelian  $G_2$  group on the vacuum exterior must act orthogonally transitively [9]. In the works on global models for isolated rotating bodies in equilibrium, and specifically those just mentioned, a  $G_2$  orthogonally transitive (OT) has been always assumed *also* on the interior region. Denoting by  $\vec{\xi}^I$  and  $\vec{\eta}^I$  two independent vector fields generating the  $G_2$  at the interior, orthogonal transitivity is equivalent to

$$\star(\xi^I \wedge \eta^I \wedge d\eta^I) = \star(\xi^I \wedge \eta^I \wedge d\xi^I) = 0, \quad (1)$$

where the star stands for the Hodge dual. This is the so-called circularity condition, and is equivalent to the absence of convective motions in fluids without energy flux [10]. Abusing the terminology, in the following non-convective will refer to OT and vice versa. This is, of course, a very restrictive condition, since astrophysical objects are likely to naturally include non-circular motions. In fact, recent studies on relativistic stars such as magnetars already take into account the existence of toroidal fields or meridional flows [11] (and references therein).

It has been recently shown [12] that since the  $G_2$  group is preserved through the matching and it is OT in the exterior, (1) will necessarily hold on the matching hypersurface  $\Sigma$  (see (11) below). Once this result was proved, the remaining question was whether (11) exhausts all the restrictions on the  $G_2$  group in the interior or not.

Herein, the role of convective motions is studied by performing the matching with the most general interiors allowed by the present problem. The result presented here shows how the condition found in [12] is the only extra condition that appears when generalising to a non-OT  $G_2$  on the interior region, and hence, how the non-OT terms (convective components of the motions, in particular) have no direct effect on the exterior field nor on the external shape of the body. Clearly, this exhibits non-uniqueness of isolated rotating bodies generating the same exterior field, under arbitrary addition of convective components that vanish on the boundary of the body. On the other hand, this makes the aforementioned results [2, 3, 4] applicable also in the general case.

‡ For studies on the local problem see also [5] and [6] and references therein. For accounts on the two problems separately the reader is referred to [7] and [8], references therein and references in [2].

It must be stressed that, of course, any assumption on the matter model would relate convective components to other quantities and hence, in an indirect manner, eventually affect the shape of the body and the exterior field. Note that no consideration on the matter model is made in this work.

## 2. The matching conditions for a general interior

When (1) is not imposed on the interior region  $(\mathcal{V}_I, g_I)$ , allowing then for convective interiors, there exists a coordinate system  $\{T, \Phi, r, \zeta\}$  in which the line-element reads [13]

$$ds_I^2 = -e^{2V} (dT + Bd\Phi + W_2 d\zeta)^2 + e^{-2V} [e^{2h} (dr^2 + d\zeta^2) + \alpha^2 (d\Phi + W_3 d\zeta)^2], \quad (2)$$

where  $V, B, h, W_2, W_3$  and  $\alpha$  are functions of  $r$  and  $\zeta$  and  $\vec{\eta}^I = \partial_\Phi$  is the axial Killing vector. The stationary Killing can be chosen to be  $\vec{\xi}^I = \partial_T$ . Conditions (1) are explicitly given by

$$W_{3,r} = 0, \quad (BW_3 - W_2)_{,r} = 0. \quad (3)$$

Under these conditions a coordinate change exists which leaves the Killing vectors  $\vec{\xi}^I$  and  $\vec{\eta}^I$  invariant and sets  $W_3 = W_2 = 0$  in (2). The metric for the exterior *vacuum* region  $(\mathcal{V}_E, g_E)$ , assumed to be free of ergoregions and/or Killing horizons, can always be cast in the following form using the so-called Weyl coordinates

$$ds_E^2 = -e^{2U} (dt + Ad\phi)^2 + e^{-2U} [e^{2k} (\rho^2 dz^2) + \rho^2 d\phi^2], \quad (4)$$

where  $U, A, k$  are functions of  $\rho$  and  $z$ . The axial Killing vector is given by  $\vec{\eta}^E = \partial_\phi$  and the axis of symmetry is located at  $\rho = 0$ . The coordinate  $t$  can be chosen to have an intrinsic meaning, namely to measure proper time of an observer at infinity, and hence the Killing vector  $\vec{\xi}^E = \partial_t$  is unit at infinity. With this choice, the remaining coordinate freedom in (4) consists only of constant shifts of  $t, \phi$  and  $z$ .

The matching preserving the stationarity and axial symmetry is performed then as in [2] (see also [12]), to which the reader is referred to for details. Local coordinates  $\{\tau, \varphi, \lambda\}$  can be chosen in an abstract hypersurface  $\sigma$  such that the embedding  $\chi_E : \sigma \rightarrow \mathcal{V}$  is given by

$$\chi_E : \{t = \tau, \phi = \varphi, \rho = \rho(\lambda), z = z(\lambda)\},$$

so that the images of  $\partial_\tau$  and  $\partial_\varphi$  by  $d\chi_E$  correspond to  $\partial_t|_{\chi_E(\sigma)} (\equiv \vec{e}_1^E)$  and  $\partial_\varphi|_{\chi_E(\sigma)} (\equiv \vec{e}_2^E)$  respectively. The image of the third vector  $\partial_\lambda$ , namely  $\vec{e}_3^E \equiv \dot{\rho}\partial_\rho + \dot{z}\partial_z|_{\chi_E(\sigma)}$  where the dot indicates derivative with respect to  $\lambda$ , has been chosen orthogonal to  $\vec{e}_1^E$  and  $\vec{e}_2^E$ . Regarding the embedding for the interior, the fact that the axial symmetry has an intrinsic meaning imposes  $d\chi_I(\partial_\varphi|_\sigma) = \partial_\Phi|_{\chi_I(\sigma)} (\equiv \vec{e}_2^I)$ . At this point the symmetry-preserving matching introduces two parameters,  $a$  and  $b$ , by allowing

$$d\chi_I(\partial_\tau|_\sigma) = a(\partial_T + b\partial_\Phi)|_{\chi_I(\sigma)}.$$

The linear coordinate change in  $(\mathcal{V}_I, g_I)$

$$\Phi = \Phi' + abT', \quad T = aT', \quad (5)$$

which implies  $\partial'_T = a(\partial_T + b\partial_\Phi)$  and  $\partial'_\Phi = \partial_\Phi$ , is useful to deal with this freedom, since it keeps (2) (with primes) and leaves invariant the axial Killing vector. Substituting unprimed by primed quantities in (2), the new metric functions read

$$\alpha' = a\alpha, \quad h' - V' = h - V, \quad (6)$$

$$e^{2V'} = a^2 \left[ (1+bB)^2 e^{2V} - \alpha^2 b^2 e^{-2V} \right], \quad B' = \frac{B(1+bB)e^{2V} - \alpha^2 b e^{-2V}}{a[(1+bB)^2 e^{2V} - \alpha^2 b^2 e^{-2V}]}, \quad (7)$$

$$W'_2 = \frac{W_2(1+bB)e^{2V} - \alpha^2 b W_3 e^{-2V}}{a[(1+bB)^2 e^{2V} - \alpha^2 b^2 e^{-2V}]}, \quad W'_3 = W_3(1+bB) - bW_2. \quad (8)$$

The functions  $W_2$  and  $W_3$  only appear in  $W'_2$  and  $W'_3$ , and therefore the expressions for  $\alpha'$ ,  $h'$ ,  $V'$  and  $B'$  in the OT case are exactly these very ones (see [2]).

Now one can use the new coordinate system (5) by dropping primes everywhere. Nevertheless, it must be noticed that if the interior  $(\mathcal{V}_I, g_I)$  is explicitly given, then the freedom introduced by the matching through (6)-(8) must be taken into account. In these new coordinates we obtain  $d\chi_I(\partial_\tau|_\sigma) = \partial_T|_{\chi_I(\sigma)} (\equiv \vec{e}_1^I)$  together with  $d\chi_I(\partial_\varphi|_\sigma) = \partial_\Phi|_{\chi_I(\sigma)} (\equiv \vec{e}_2^I)$  [2]. The continuity of the first fundamental form forces the image of  $\partial_\lambda$ , namely  $\vec{e}_3^I$ , to be orthogonal to  $\vec{e}_1^I$  and  $\vec{e}_2^I$ , so that  $\vec{e}_3^I = \dot{r}\partial_r + \dot{\zeta}[(BW_3 - W_2)\partial_T - W_3\partial_\Phi + \partial_\zeta]|_{\chi_I(\sigma)}$ . Therefore, the most general form of the embedding  $\chi_I : \sigma \rightarrow \mathcal{V}_I$  reads

$$\chi_I : \{T = \tau + f_T(\lambda), \Phi = \varphi + f_\Phi(\lambda), r = r(\lambda), \zeta = \zeta(\lambda)\},$$

where  $\dot{f}_T = (BW_3 - W_2)|_\sigma \dot{\zeta}$  and  $\dot{f}_\Phi = -W_3|_\sigma \dot{\zeta}$ . The four functions  $\rho(\lambda), z(\lambda), r(\lambda)$  and  $\zeta(\lambda)$  define then the matching hypersurface  $\Sigma \equiv \chi_I(\sigma) = \chi_E(\sigma)$  and will be determined, as shown below, by the matching conditions. The explicit expressions of the (non-unit)§ normal forms to the matching hypersurface are  $\mathbf{n}^I = e^{2h}(\dot{\zeta}dr + \dot{r}d\zeta)|_\Sigma$  and  $\mathbf{n}^E = e^{2k}(\dot{z}d\rho + \dot{\rho}dz)|_\Sigma$ .

In the OT case, once the interior metric  $g_I$  is known, the whole matching conditions were shown in [14, 2] to reorganise into three sets of conditions: namely, (a) a set of conditions on the interior hypersurface, (b) a set of conditions for the exterior hypersurface, and (c) boundary conditions for the exterior problem.

In the present general case the three sets are recovered again. Indeed, the whole set of matching conditions can be cast in the following form:

(a) *Conditions on the interior hypersurface:*

$$n^{I\alpha} n^{I\beta} S_{\alpha\beta}|_\Sigma = 0, \quad n^{I\alpha} \vec{e}_3^{I\beta} S_{\alpha\beta}|_\Sigma = 0, \quad (9)$$

$$W_{3,r}|_\Sigma = 0, \quad (BW_3 - W_2)_{,r}|_\Sigma = 0, \quad (10)$$

where  $S_{\alpha\beta}$  is the Einstein tensor in the interior region. The two new equations (10) are equivalent to (see (3))

$$\star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\eta}^I)|_\Sigma = \star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\xi}^I)|_\Sigma = 0, \quad (11)$$

§ The only necessary requirement for the matching is that they have the same norm and relative orientations.

as expected, since they were already shown in [12] to be necessary conditions (even in more general settings). Importantly, decomposing the Einstein tensor at the interior into its OT and non-OT part, so that  $S_{\alpha\beta} = S_{\alpha\beta}^{OT} + S_{\alpha\beta}^W$  defining  $S_{\alpha\beta}^{OT} \equiv S_{\alpha\beta}|_{\xi^I \wedge \eta^I \wedge d\eta^I = \xi^I \wedge \eta^I \wedge d\xi^I = 0}$ , then equations (9) are equivalent to

$$n^{I\alpha} n^{J\beta} S_{\alpha\beta}^{OT}|_{\Sigma} = 0, \quad n^{I\alpha} e_3^{J\beta} S_{\alpha\beta}^{OT}|_{\Sigma} = 0, \quad (12)$$

provided that (10) is satisfied. Although the decomposition is not invariantly defined, the contractions appearing in equations (12) are.|| Therefore, the system composed by (9) and (10) is equivalent to (12) and (10). Equations (12) constitute the set (a) in [2].

Equations (9)-(10) form an overdetermined system of four ordinary differential equations for  $r(\lambda)$  and  $\zeta(\lambda)$ . If a solution exists, then the matching will be possible. Generically, the interior matching hypersurface will be uniquely determined. Nevertheless, there are cases where (9)-(10) contain no information, and hence the matching is possible across any timelike hypersurface preserving the symmetry.

As mentioned, the difference between the OT and the non-OT cases lies in the fulfilment of equations (3). Conditions (10) state that (3) hold on the matching hypersurface. At this point only the normal derivatives of (3) could make a difference with respect to the OT case in the rest of the matching conditions. Surprisingly, after a straightforward calculation, these normal derivatives can be shown not to appear. Indeed, the rest of the matching conditions are exactly the same as in the OT case [2]. These are the following:

(b) *Exterior matching hypersurface*: Once the interior hypersurface has been determined, the functions defining the exterior matching hypersurface  $\rho(\lambda)$  and  $z(\lambda)$  are uniquely determined by

$$\rho(\lambda) = \alpha|_{\Sigma}, \quad \dot{z}(\lambda) = \alpha_{,r}\dot{\zeta} - \alpha_{,\zeta}\dot{r}|_{\Sigma}. \quad (13)$$

Note that an additive constant in  $z(\lambda)$  is spurious due to the freedom  $z \rightarrow z + const.$

(c) *Boundary conditions for the exterior problem*: The rest of the matching conditions read

$$\begin{aligned} U|_{\Sigma} &= V|_{\Sigma}, & A|_{\Sigma} &= B|_{\Sigma}, \\ \vec{n}^E(U)|_{\Sigma} &= \vec{n}^I(V)|_{\Sigma}, & \vec{n}^E(A)|_{\Sigma} &= \vec{n}^I(B)|_{\Sigma}. \end{aligned} \quad (14)$$

These four equations provide the boundary conditions on the metric functions  $U$  and  $A$ , which translate into overdetermined boundary conditions on the Ernst potential for the exterior vacuum problem. To be more precise, the boundary conditions leave still a degree of freedom in the form of an additive constant in the twist potential (see [2]). Nevertheless, as shown in [2], if a solution exists, then this additive constant is fixed and thus the Ernst potential is determined everywhere in the exterior region. The remaining

|| As a matter of fact, one always has  $n^{I\alpha} e_3^{J\beta} S_{\alpha\beta}^W|_{\Sigma} = 0$  and  $n^{I\alpha} n^{J\beta} S_{\alpha\beta}^W|_{\Sigma} = \frac{\vec{e}_3^2 \vec{\xi}^2}{4[(\vec{\xi} \cdot \vec{\eta})^2 - \vec{\xi}^2 \vec{\eta}^2]^2} \left[ \star(\xi^I \wedge \eta^J \wedge d\xi^I) \vec{\eta} + \star(\xi^I \wedge \eta^J \wedge d\eta^I) \vec{\xi} \right]^2|_{\Sigma}$ , with the obvious notation  $\vec{v}^2 \equiv v^\alpha v_\alpha$ .

function  $k$  in (4) is found, up to an additive constant, by quadratures. This constant is fixed by the matching procedure by using the complementary equation

$$k|_{\Sigma} = \left[ h - \frac{1}{2} \ln (\alpha_{,r}^2 + \alpha_{,\zeta}^2) \right] \Big|_{\Sigma}. \quad (15)$$

This matching condition is complementary in the sense that the previous equations (9)-(10) ensure that its derivative with respect to  $\lambda$  is satisfied, and hence (15) only determines the additive constant in  $k$ . This comment was incidentally left out in [2].

### 2.1. Rewriting the set (a)

Conditions (11) are purely geometrical. In [14], two matching conditions were rewritten in terms of the Einstein tensor (so-called Israel conditions [15]) and the analogous procedure has been used here to get (9). The question that arises is whether (10) are also equivalent to any of the Israel conditions, since that would relate the geometrical aspect of (11) with physical properties of the interior matter content on  $\Sigma$ . Let me stress here the fact that the Israel conditions are consequences of the matching conditions, and hence, necessary (not sufficient in general) conditions for the matching. Since we have a vacuum exterior, the Israel conditions read

$$\begin{aligned} & \text{(i)} \quad n^{I\alpha} n^{I\beta} S_{\alpha\beta} |_{\Sigma} = 0, \quad \text{(ii)} \quad n^{I\alpha} e_3^{I\beta} S_{\alpha\beta} |_{\Sigma} = 0, \\ & \text{(iii)} \quad n^{I\alpha} e_1^{I\beta} S_{\alpha\beta} |_{\Sigma} = 0, \quad \text{(iv)} \quad n^{I\alpha} e_2^{I\beta} S_{\alpha\beta} |_{\Sigma} = 0. \end{aligned} \quad (16)$$

In the OT case the two last relations (iii) and (iv) are identically satisfied because of the structure of the Einstein tensor inherited by the OT  $G_2$ . Therefore, in the OT case the two equations (9) constitute the whole set of non-trivial Israel conditions, and thus the set (a) of matching conditions is made up by the Israel conditions. But in the general case the two last relations in (16) are non-trivial (of course, they will be satisfied once the whole set of matching conditions (9),(10),(13),(14) is satisfied).

With the help of a well known identity [9, 10] one can find

$$\begin{aligned} n^{I\alpha} e_1^{I\beta} S_{\alpha\beta} |_{\Sigma} &= - \frac{e^{2V}}{2\alpha} \Bigg|_{\Sigma} \frac{d}{d\lambda} \left[ \star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\xi}^I) \right] |_{\Sigma}, \\ n^{I\alpha} e_2^{I\beta} S_{\alpha\beta} |_{\Sigma} &= - \frac{e^{2V}}{2\alpha} \Bigg|_{\Sigma} \frac{d}{d\lambda} \left[ \star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\eta}^I) \right] |_{\Sigma}, \end{aligned} \quad (17)$$

on the matching hypersurface. From these identities it readily follows that (11) imply the Israel conditions (iii) and (iv), as mentioned. On the other hand, and more interestingly, if (iii) and (iv) hold, then  $\star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\xi}^I)|_{\Sigma}$  and  $\star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\eta}^I)|_{\Sigma}$  are constants on  $\Sigma$ . Now, if our interior region is to describe a spatially compact and simply connected object  $\Sigma$  will intersect the axis of symmetry. At those points  $\boldsymbol{\eta}^I$  vanishes, and thus  $\star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\xi}^I)|_{\Sigma}$  and  $\star(\boldsymbol{\xi}^I \wedge \boldsymbol{\eta}^I \wedge d\boldsymbol{\eta}^I)|_{\Sigma}$  will vanish. This argument involving the axis is analogous to that used to show that the exterior region must admit a OT  $G_2$  [9, 10]. Therefore, in the cases we will be interested in, the Israel conditions (iii) and (iv) are equivalent to the conditions (11). The relation between the geometrical properties of the  $G_2$  at the interior with the properties of the matter content on the boundary of the body is then manifest.

### 3. Summary and conclusions

The complete set of matching conditions for the general case can be cast into the three sets (a), (b) and (c), as described above, in analogy with the OT case. In fact, the only difference with the OT case lies in the set (a), where now we have two more equations (11). Notice also that although there might be hidden non-OT terms in (9), this is not the case, as (9) are equivalent to (12).

In subsection 2.1 it has been shown that for spatially compact and simply connected interiors (9)-(10) are equivalent to (16), and therefore the set (a) of conditions is equivalent to the complete set of Israel conditions. This is analogous to what happens in the non-convective case, the difference being that the Israel conditions in the present general case constitute four relations instead of only two.

The rest of the conditions (sets (b) and (c)), which concern the unknown exterior region, are the same as in the OT case. This fact is important, because it demonstrates that the exterior problem is “independent” of any convective motions in the interior. More precisely, once the non-OT terms have been proven to vanish on the boundary of the body, see (11), there is no other explicit information coming from the inner non-OT terms affecting the boundary conditions (nor the exterior boundary itself) on the Ernst potential for the exterior problem. Therefore, all the existing results and studies on the existence and uniqueness of asymptotically flat vacuum exteriors such as [14, 2, 3, 4], where the circularity condition has been always assumed, apply equally in the general case.

In particular, the existence of an asymptotically flat vacuum exterior has been shown to impose a number of conditions on the overdetermined boundary data for the Ernst potential [4]. Once we have an interior region “shaped” by the set (a), those conditions translate onto the interior quantities through (14) together with (13), becoming the conditions our interior has to satisfy in order to describe a truly isolated body. The point made here implies that the existence of an asymptotically flat vacuum exterior poses no conditions onto the possible convective motions inside the body.

On the other hand, we know that the exterior field generated by a non-convective interior region describing an isolated rotating body in equilibrium is unique<sup>¶</sup> [2]. Now, given a global model composed by an OT interior together with its corresponding asymptotically flat exterior, so that (12), (13), (14) (and (15)) hold on, say,  $\Sigma_{OT}$ , one can always introduce arbitrary convective components vanishing on  $\Sigma_{OT}$  and thus generate different interiors keeping the same shape and exterior fields. Conversely, given a general interior explicit metric  $g_I$  such that the set (a) of conditions is satisfied for some hypersurface  $\Sigma^I$ , the exterior (if it exists) will be unique and the same as the one generated by  $g_I|_{W_2=W_3=0}$ .

<sup>¶</sup> Once the identification of the interior with the exterior through  $\Sigma$  has been prescribed by fixing  $a$  and  $b$ .

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